On the Origin of the Lorentz Transformation

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Abstract

The Lorentz Transformation, which is considered as constitutive for the Special Relativity Theory, was invented by Voigt in 1887, adopted by Lorentz in 1904, and baptized by Poincaré in 1906. Einstein probably picked it up from Voigt directly.

1 Maxwell's wave equation

Maxwell's ether theory of light was developed from his first order equations and resulted in a homogeneous wave equation for the vector potential:

$$c^2 \, \Delta \vec{A} = \frac{\partial^2 \vec{A}}{\partial t^2}$$

The electromagnetic field could be calculated by differentiating this potential with respect to time and space: $\vec{E} = -\partial \vec{A}\big/\partial t$, $\vec{B} = \operatorname{rot} \vec{A}$. For a wave polarized in y–direction, e.g., and travelling in x–direction one may write for the y–component of $\vec{A}$:

$$c^2 \frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 A}{\partial t^2} \tag{1}$$

Formally this equation is the same as a wave equation for sound, e.g.

$$c_S^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2} \tag{2}$$

where c_S is the sound velocity in air and p is the pressure perturbation. Maxwell took the propagation velocity in (1) as the velocity of light with respect to the ether that was perceived by him as a hypothetical medium in which light propagates like sound in air.

From the formal similarity of (1) and (2) follow similar solutions, e.g. plane waves:

$$A = A_0 \, \cos\left(k \, x - \omega \, t\right) \tag{3}$$

At a fixed time t the wave has an oscillatory behaviour in space with wavelength $\lambda = 2\pi/k$, and at a fixed point x the amplitude A of the wave oscillates with frequency $\nu = \omega/2\pi$. A point of fixed phase, e.g. $k \, x - \omega \, t = 0$, travels with phase velocity

$$\frac{x}{t} = \frac{\omega}{k} = c \tag{4}$$

This result is easily obtained by inserting (3) into (1).

If an observer travels along the x–axis, i.e. parallel to the $\vec{k}$–vector, with velocity v, we have the obvious Galilei connection between his coordinates $(x', \, t')$ and the wave coordinates:

$$x' = x - v \, t \,, \quad t' = t \tag{5}$$

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Substituting this into (3) one obtains again a plane wave in the moving system:

$$A = A_0 \cos\left(k \left(x' + v\, t'\right) - \omega\, t'\right) = A_0 \cos\left(k\, x' - (\omega - k\, v)\, t'\right) \qquad (6)$$

The phase velocity as observed in that system is:

$$\frac{x'}{t'} = \frac{\omega}{k} - v = c - v \qquad (7)$$

and the angular frequency becomes

$$\omega' = \omega - k\, v = \omega\left(1 - v/c\right) \qquad (8)$$

according to (6) and (4). This equation reflects the Doppler Effect which is well known from observations both in sound and in light.

Due to the change of the phase velocity (7) it is clear that equation (1) must change its form, when transformed into a moving system. Using the chain rule for differentiation we have from eq. (5):

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial}{\partial x'}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = -v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}\,, \qquad \frac{\partial^2}{\partial t^2} = v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial^2}{\partial t'^2} \qquad (9)$$

Substituting these expressions into (1) yields the wave equation in the moving system:

$$c^2\frac{\partial^2 A}{\partial x'^2} = v^2\frac{\partial^2 A}{\partial x'^2} - 2v\frac{\partial^2 A}{\partial x'\partial t'} + \frac{\partial^2 A}{\partial t'^2} \qquad (10)$$

Inserting the plane wave ansatz $\cos\left(k\, x' - \omega'\, t'\right)$, one obtains the dispersion relation

$$\left(c^2 - v^2\right) k^2 = 2v\, k\, \omega' + \omega'^2 \qquad (11)$$

in agreement with (4) and (8).

2 Voigt's theory of the Doppler Effect and its reception

Although the theory developed so far describes very well the Doppler Effect both for sound and light, Woldemar Voigt published an alternative theory for elastic media in 1887 which he called *On Doppler's Principle*[1]. He insisted that equation (1) should maintain its form upon transformation into a moving system, namely equation (10) should read:

$$c^2\frac{\partial^2 A}{\partial x'^2} = \frac{\partial^2 A}{\partial t'^2} \qquad (12)$$

This amounts to requiring that the phase velocity of a wave is independent of the motion of the observer. It is unclear how Voigt arrived at this curious idea.

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[1]W. Voigt, *Ueber das Doppler'sche Princip,* Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg–Augusts–Universität zu Göttingen, No. 2, 10. März 1887

He simply commented eq. (12) with the cryptic remark "*da ja sein muss*", "as it must be". We cannot exclude that he was mistaken by a misconception of the nature of propagating waves. In any case, in order to achieve his goal he transformed time and made it a linear function of space. Instead of (5) he wrote

$$x' = x - v\,t\;, \quad t' = t - x\,v\big/c^2 \tag{13}$$

Now the phase velocity in the moving system becomes with (13):

$$c' = \frac{x'}{t'} = \frac{x - v\,t}{t - x\,v\big/c^2} \tag{14}$$

Inserting $x = c\,t$ from (4) one obtains instead of (7):

$$c' = \frac{c\,t - v\,t}{t - t\,v/c} = c \tag{15}$$

Voigt did apparently not realize that the independence of the phase velocity from the motion of the observer is in blatant contradiction to the observations in sound for which his theory was supposed to hold as well. One can hardly accept that time should transform according to (13) in compliance with all the different wave velocities occurring in various elastic media.

For reasons we do not know either, Voigt's transformation appeared attractive to Lorentz in his pursuit to develop an electrodynamics for moving media. In his paper of 1904[2] he only says: "I take as new independent variables..." and introduces subsequently Voigt's transformation (13) without mentioning his name. In a monograph of 1913[3], however, which contained early papers related to relativity by Lorentz, Einstein, and Minkowski, he added a footnote where he acknowledged the equivalence of his transformation to that of Voigt's from 1887.

In his paper of 1906[4] (left out in Ref. 3) Poincaré referred not to Voigt's, but to Lorentz's work and gave the transformation (13) Lorentz's name. It appears that he was intrigued by the postulated constancy of the light velocity for all observers regardless of their motion. This seemed to be in accordance with his general principle of relativity that would make it impossible to distinguish different inertial states of motion by electromagnetic measurements. Poincaré also recognized that the linear relationship (13) could be multiplied by a constant factor which cancels in (14), thus keeping the velocity of light still constant. The final form of the "Lorentz Transformation" took then the symmetrical shape:

$$x' = \gamma\,(x - v\,t)\;, \; y' = y\;, \; z' = z\;, \; t' = \gamma\left(t - x\,v\big/c^2\right)\;, \; \gamma = 1\Big/\sqrt{1 - v^2\big/c^2} \tag{16}$$

[2]H. A. Lorentz, *Electromagnetic phenomena in a system moving with any velocity smaller than that of light,* Proceedings Acad. Sc. Amsterdam **6** (1904) 809

[3]H. A. Lorentz, A. Einstein, H. Minkowski, *Das Relativitätsprinzip, eine Sammlung von Abhandlungen,* ed. by O. Blumenthal, B. G. Teubner Verlag , Stuttgart (1913)

[4]H. Poincaré, *Sur la dynamique de l'électron*, Rendiconti del Circolo matematico di Palermo, **21** (1906) 129

which is form-invariant to the inverse transformation choosing $v \to -v$:

$$x = \gamma\left(x' + v\,t'\right) , \quad y = y', \quad z = z', \quad t = \gamma\left(t' + x'\,v/c^2\right) \qquad (17)$$

It remains to demonstrate that application of (16) transforms indeed eq. (1) into eq. (12). Instead of (9) we have from (16):

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = \gamma\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right) \qquad (18)$$

$$\frac{\partial^2}{\partial x^2} = \gamma^2\left(\frac{\partial^2}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2}{\partial x'\partial t'} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2}\right)$$

$$\frac{\partial^2}{\partial t^2} = \gamma^2\left(v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial^2}{\partial t'^2}\right)$$

If one substitutes these expressions into (1), one obtains:

$$c^2\frac{\partial^2 A}{\partial x'^2}\gamma^2\left(1 - v^2/c^2\right) = \frac{\partial^2 A}{\partial t'^2}\gamma^2\left(1 - v^2/c^2\right) \qquad (19)$$

where the γ-factor cancels.

3 Einstein's *procedere*

Einstein like Poincaré was convinced of the validity of the generalized relativity principle. In his famous paper of 1905[5] he raised this conjecture to a postulate and claimed that he could derive the Lorentz Transformation by using a second postulate, namely "that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body". A careful analysis of § 3 of his paper shows, however, that he worked on the same assumption as Voigt, namely that the velocity of light is constant for any moving observer. This is reflected in the little relationship $^{1}\!/_{2}\left(\tau_0 + \tau_2\right) = \tau_1$ on page 898. It refers to a situation where a light signal is emitted from the origin of a moving coordinate system at time point τ_0, travels towards some point on the x–axis where it arrives at time τ_1, and returns to the origin upon reflection at time τ_2. Einstein uses this inconspicuous relationship in order to express in a somewhat convoluted way his assumption $c = $ const in the moving system as a precondition for the derivation. The technical term for taking the conclusion to be proved into the premise is called *petitio principii*. Anyhow, the first postulate, namely his "principle of relativity", is not used in Einstein's derivation of the Lorentz Transformation.

One cannot exclude that Einstein, who did not quote anyone in Ref. 5, found his transformation directly in Voigt's paper of 1887. In the monograph of 1913 (Ref. 3) he assures us that he was not aware of Lorentz's paper of 1904. This is credible, as he did not adopt Lorentz's nomenclature $(x, y, z, t) \leftrightarrow$

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[5] A. Einstein, *Zur Elektrodynamik bewegter Körper,* Ann. d. Phys., **17** (1905) 891

(x', y', z', t'), but he used Voigt's: $(x, y, z, t) \leftrightarrow (\xi, \eta, \zeta, \tau)$. He also added a footnote where he stated that the "Lorentz Transformation" is much easier to be derived using Voigt's postulate, but – in contrast to Lorentz – he avoided actually to mention his name.

It is instructive to recognize that Voigt's absurd claim of the constancy of a wave's phase velocity makes as little sense in electrodynamics as it does in acoustics. The linear Doppler Effect as described in (8) is normally thought to be due to a variation of an observer's velocity with respect to the wave crests. For a wave we have the well known relationship:

$$c = \lambda \nu \tag{20}$$

When we measure an increased frequency ν', more wave crests hit the detector per unit time, because the detector has an increased relative velocity $c + v$ with respect to the wave fronts. If one claims, however, that the relative velocity between detector and wave front is always c, one must conclude from (20) and (8) that the wavelength transforms linearly with the velocity:

$$\frac{\lambda'}{\lambda} = \frac{\nu}{\nu'} = \frac{c}{c \pm v} \tag{21}$$

This relationship is not often discussed in textbooks on SRT, probably because of its absurd consequences. If the length of light waves would transform linearly with velocity, but the distance between mirrors in a resonator transforms quadratically, as the Lorentz contraction assumes, a laser would be extinguished, simply because an observer passes by at high velocity, thereby destroying the resonance condition which requires an integral number of modes in the resonator. It is hard to take this as a physical fact. One uses instead the linear Doppler Effect to determine the velocity of the earth relative to the wave crests of the cosmic background radiation and finds 370 km/s versus τ Leonis, in obvious violation of the generalized relativity principle.

Concluding this little essay I would like to communicate an interesting aperçu. According to Gerald Holton[6] Einstein kept denying that the Michelson-Morley experiment had any influence on the development of his theory. Probably he had read a paper with the title *Theory of light for moving media* by W. Voigt[7] which appeared two months later than Ref. 1 in the same Journal. On page 233 Voigt analyzes the Michelson experiment and comes to the conclusion: „Hieraus folgt, daß die Beobachtungsresultate bei der Anordnung des Experimentes, wie sie Herr M i c h e l s o n gewählt hat, v o n d e r T r a n s l a t - i o n v ö l l i g u n a b h ä n g i g s i n d, daß also Herr M i c h e l s o n die negativen Resultate, die er factisch erhalten hat, erhalten m u ß t e, gleichviel,

[6]G. Holton, EINSTEIN, MICHELSON, AND THE *"CRUCIAL" EXPERIMENT*, American Journal of Physics **37** (1969) 968

[7]W. Voigt, *Theorie des Lichts für bewegte Medien,* Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg–Augusts–Universität zu Göttingen, No. 8, 11. Mai 1887

ob sich der Aether mit der Erde bewegt oder nicht." [8]

If Einstein believed that Voigt was right – and we do not have any evidence for the contrary – he could not possibly take Michelson's result as experimental confirmation for the correctness of his theory. So, it suggests itself that at some point in his investigations he had become aware of Voigt's article (Ref. 7) that had appeared in a renowned Journal. And if this was the case, he could hardly have overlooked Voigt's paper (Ref.1) which had been published two months before. Possibly, it offered him the transformation he could exploit for his purpose.

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[8] "It follows that the observed results in the experimental set-up that was chosen by Mr. Michelson a r e e n t i r e l y i n d e p e n d e n t o f a n y t r a n s l a t i o n. Hence, Mr. Michelson obtained the negative results which he factually obtained with n e c e s s i t y, regardless whether the ether moves with the earth or not."